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A NEW UPPER BOUND FOR THE LENGTH OF SNAKES

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A d-dimensional circuit code of spread s is a simple circuit C in the graph of the d-dimen sional unit cube with the property that for any vertices x and y of C which differ in exactly r coordinates, $r<s$, there exists a path from x to y consisting of r edges of C. This property is useful for detecting and limiting errors. In this paper we give a new upper bound for the maximum length of a d-dimensional circuit code of spread 2.

1. Introduction and definitions

We denote by $I(d)$ the graph with the various d-tuples of binary digits as vertices, two vertices being adjacent if and only if they differ in exactly one coordinate. Thus $I(d)$ is the graph of the d-dimensional unit-cube. If d is understood, we write I instead of $I(d)$. The set of vertices of a graph G will be denoted by $V(G)$, and, for $x, y \in V(G)$, $d_G(x, y)$ will be the minimum number of edges forming a path from x to y. (In this paper we are only concerned with connected graphs, therefore such a path always exists.) The subgraphs of $I(d)$ are called *d-dimensional codes.* In this paper we are interested in a special class of d-dimensional codes, namely d-dimensional codes of spread s.

Definition 1. Let C be a simple circuit in $I(d)$, and $s \in \mathbf{N}$. If for all vertices x and y of C

$$d_{I(d)}(x, y) \geqq \min \{d_C(x, y), s\},$$

then C is called a *d-dimensional circuit code of spread s.*

Circuit codes of spread 1 are commonly called Gray-codes, and obviously every circuit in $I(d)$ is a Gray-code. Circuit codes of spread 2 were introduced by Kautz in [5], who called them *unit-distance error-checking codes* or *snake-in-the-box-codes.* For the use of such codes e.g. in analog-to-digital-conversion see Klee in [6]. A circuit code of spread 2 is nothing else than a chordless circuit in the graph $I(d)$. Usually $C(d, s)$ denotes the length of a longest d-dimensional circuit code of spread s. In this paper we are interested in upper bounds for $C(d, 2)$. The problem

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of finding an upper bound for $C(d, 2)$ was first solved by Kautz in [5], who showed

$$C(d, 2) \leqq \frac{d}{d-1} 2^{d-1}.$$

For $d \geqq 4$ Abbott [1], Danzer and Klee [2], Glagolev [4], and Singleton [8] independently improved this to

$$C(d, 2) \leqq 2^{d-1}.$$

Larman [7] proved in 1968

$$C(d, 2) \leqq 2^{d-1} - 2^{d-5} d^{-6} \quad \text{for} \quad d \geqq 5,$$

from which one can conclude $C(5, 2) = 14$. Also in 1968 Douglas [3] reduced the bound to

$$C(d, 2) \leqq 2^{d-1} - \frac{2^d - 12}{7d(d-1)(d-1)+2} \quad \text{for} \quad d \geqq 6.$$

In this paper we will give a new improvement by showing

$$C(d, 2) \leqq 2^{d-1} - \frac{2^{d-1}}{d(d-5)+7}. \quad \text{for} \quad d \geqq 7.$$

Before doing so we must introduce some new definitions and notation rules. First we need a simple method to denote the vertices of $I(d)$.

Definition 2. Let $i_1, \ldots, i_n$ be coordinates with $1 \leqq n \leqq d$ and $i_1, \ldots, i_n \in \{1, \ldots, d\}$ pairwise different. Then we denote by $(i_1 i_2 \ldots i_n)$ the vector with a 1 at the coordinates $i_1, \ldots, i_n$ and a 0 at the other coordinates. (0) denotes the d-tuple with a 0 at all coordinates.

For example $(25) = (0, 1, 0, 0, 1, 0, 0)$ in $I(7)$.

The following examinations are based on the property of a circuit C of spread 2 that for all vertices x of C exactly two neighbours of x also belong to C. Thus appearing of x in C makes appearing in C impossible for most of its neighbours. The next definition will help us to describe this property more formally.

Definition 3. Let C be a d-dimensional circuit code of spread 2. Let $x \in V(C)$ and $y \in V(I)$.

(a) $Ny = \{z \in V(I) \colon d_I(y, z) = 1\}$ denotes the set of all neighbours of y in I.

(b) *x blocks y* if $y \in Nx$ and $y \notin V(C)$.

(c) *x blocks y with the weight $\frac{1}{n}$*, $n \in \mathbf{N}$, if x blocks y and $n = \operatorname{card}\left(\{z \in V(C) \colon d_I(z, y) = 1\}\right)$.

(d) We denote $Sx = \{z \in V(I) \colon x \text{ blocks } z\}$.

Example. In $I(4)$ let $C = \big((0), (1), (12), (123), (23), (234), (24), (4);\ (0)\big)$ as shown in figure 1. C is of spread 2 in $I(4)$, and e.g. we get $S(12) = \{(2), (124)\}$, $N(0) = \{(1), (2), (3), (4)\}$, $S(0) = N(0) \setminus \{(1), (4)\}$, and (0) blocks (3) with the weight $\frac{1}{2}$, whereas (12) blocks (2) with the weight $\frac{1}{4}$. Notice that (134) is not blocked by any vertex.

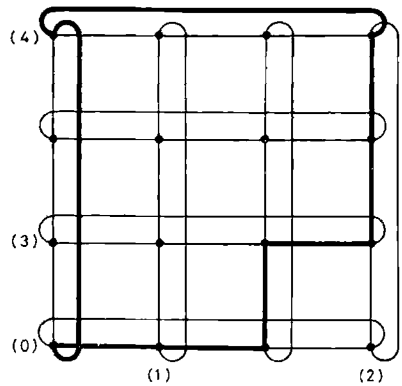

Fig. 1

Definition 4. Let C be a d-dimensional circuit code of spread 2, $x \in V(C)$. Then let g_x be the mapping from $V(I)$ to $\mathbf{Q}$ defined by

$$g_x(y) = \begin{cases} \frac{1}{n} & \text{if } x \text{ blocks } y \text{ with the weight } \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

and

$$G(x) = \sum_{y \in V(I)} g_x(y).$$

That means $G(x)$ is the sum of all weights with which x is blocking elements of $V(I)$. In our example we have $G((0)) = \frac{3}{4}$ and $G((1)) = 1$. Obviously for all $y \in V(I)$ the following equality holds

$$\sum_{x \in V(C)} g_x(y) = \begin{cases} 0 & \text{if } d_I(x, y) \geqq 2 \text{ for all } x \in V(C) \\ 0 & \text{if } y \in V(C) \\ 1 & \text{otherwise} \end{cases}$$

and we get the following important relation

(1) $$\sum_{x \in V(C)} G(x) \leqq \operatorname{card}\left(V(I) \setminus V(C)\right) = 2^d - \operatorname{card}\left(V(C)\right).$$

2. Five lemmas

Before we are able to prove the new upper bound we need five lemmas.

Lemma 1. *For* $d \geqq 7$ *let* $C = (v_1, v_2, \ldots, v_n;\ v_1)$ *be a d-dimensional circuit code of spread* 2 *with* $v_1 = (123)$, $v_2 = (12)$, $v_3 = (1)$, $v_4 = (0)$, *and* $v_5 = (3)$. *If* $\operatorname{card}\left(N(13) \cap \cap V(C)\right) = d - 1$, *then*

$$G(v_3) > \frac{d-4}{d-2} + \frac{2}{d-3}.$$

Proof. Putting $M=\{4,5,\dots,d\}$, we have $N(13)=\{(1),(123),(3)\}\cup\{(13i): i\in M\}$. Let $(13m)$ be the one element of $N(13)$ not lying in $V(C)$, $(m\in M)$. Since $d\geqq 7$ holds, there are at least three different elements $(13i)$ with $i\in M\setminus\{m\}$ in $V(C)$. Thus there exists $v_s\in V(C)$ with $v_s=(13k)$, $k\in M\setminus\{m\}$, $d_C(v_1,v_s)\geqq 4$, and $d_C(v_5,v_s)\geqq 4$. We get $Nv_s=\{(13),(1k),(3k),(123k)\}\cup\{(13kj): j\in M\setminus\{k\}\}$.

Let $y\in Nv_s\cap V(C)$. If $y=(13)$ or $y=(1k)$, then $d_I(v_3,y)=1$ and $d_C(v_3,y)\geqq 5$. If $y=(3k)$, then $d_I(v_5,y)=1$ and $d_C(v_5,y)\geqq 3$. If $y=(123k)$, then $d_I(v_1,y)=1$ and $d_C(v_1,y)\geqq 3$. All these cases are impossible since C is of spread 2. Hence $v_{s-1}=(13kj)$ with $j\in M\setminus\{k\}$ and $v_{s+1}=(13kp)$ with $p\in M\setminus\{k,j\}$. Then $j\neq m$ or $p\neq m$. Without loss of generality we assume $j\neq m$.

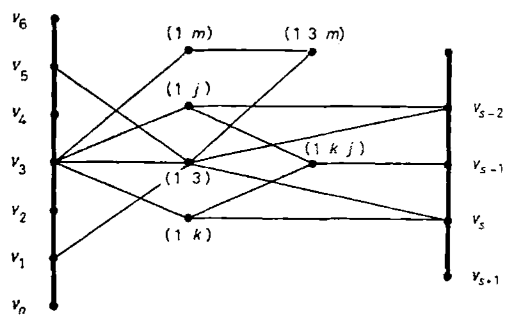

Fig. 2

From $\mathrm{card}\,(N(13)\cap V(C))=d-1$ and $(13m)\notin V(C)$ we conclude $(13j)\in V(C)$. Then $v_{s-2}=(13j)$ (otherwise we get $d_I(v_{s-1},(13j))=1$ and $d_C(v_{s-1},(13j))>1$, a contradiction). The situation is summarized in figure 2. Hence we get the following bounds:

$g_{v_3}((1k))\geqq\frac{1}{d-3}$ since $(1kj)\in S(13kj)$, $(k)\in Sv_4$, and $(12k)\in Sv_2$;

$g_{v_3}((1j))\geqq\frac{1}{d-3}$ since $(1kj)\in S(13kj)$, $(j)\in Sv_4$, and $(12j)\in Sv_2$;

$g_{v_3}((1m))\geqq\frac{1}{d-3}$ since $(13m)\notin V(C)$, $(m)\in Sv_4$, and $(12m)\in Sv_2$;

$g_{v_3}((13))=\frac{1}{d-1}$ by assumption;

$g_{v_3}((1i))\geqq\frac{1}{d-2}$ for $i\in M\setminus\{j,k,m\}$, since $(i)\in Sv_4$ and $(12i)\in Sv_2$.

From these inequalities we get

$$G(v_3)\geqq\frac{1}{d-1}+\frac{d-6}{d-2}+\frac{3}{d-3}>\frac{d-4}{d-2}+\frac{2}{d-3},$$

which proves lemma 1. ∎

Lemma 2. *Let d and C be as in lemma 1. If* card$(N(13)\cap V(C))=d$, *then*

$$G(v_3)>\frac{d-4}{d-2}+\frac{2}{d-3}.$$

Proof. Set $M=\{4,5,\dots,d\}$. Since (1), (3), and (123) are in $\{v_1,\dots,v_5\}$ and $d\geqq 7$, we know that at least four elements $(13j)$ with $j\in M$ appear in $\{v_6,\dots,v_n\}$. Set $i=\min\{j\in\mathrm{N}\colon 6\leqq j\leqq n \text{ and } v_j\in N(13)\}$. Let $v_i=(13p)$ be the first element of the vertex sequence $v_6,\dots,v_n$ which belongs to $N(13)$, $(p\in M)$. In the same way as in lemma 1 we can conclude that

$$\begin{aligned} v_{i+1}&=(13pq) \quad \text{with} \quad q\in M\setminus\{p\},\\ v_{i+2}&=(13q),\\ v_{i+3}&=(13qr) \quad \text{with} \quad r\in M\setminus\{p,q\},\\ v_{i+4}&=(13r),\\ v_{i+5}&=(13rs) \quad \text{with} \quad s\in M\setminus\{p,q,r\},\\ v_{i+6}&=(13s).\end{aligned}$$

This series cannot be carried on, because from $d\geqq 7$ we only know card$(M)\geqq 4$. What we already know is illustrated in figure 3. We get with a similar argumentation as in lemma 1

$$g_{v_3}((1i))\geqq\frac{1}{d-3} \quad \text{for} \quad i\in\{p,s\},$$

$$g_{v_3}((1i))\geqq\frac{1}{d-4} \quad \text{for} \quad i\in\{q,r\},$$

$$g_{v_3}((1i))\geqq\frac{1}{d-2} \quad \text{for} \quad i\in M\setminus\{p,q,r,s\},$$

$$g_{v_3}((13))\geqq\frac{1}{d}.$$

From these inequalities we conclude

$$G(v_3)\geqq\frac{1}{d}+\frac{d-7}{d-2}+\frac{2}{d-3}+\frac{2}{d-4}>\frac{d-4}{d-2}+\frac{2}{d-3},$$

which proves lemma 2. ∎

Lemma 3. *For* $d\geqq 7$ *let* $C=(v_1,v_2,\dots,v_n;\ v_1)$ *be a d-dimensional circuit code of spread* 2 *with* $v_1=(123)$, $v_2=(12)$, $v_3=(1)$, $v_4=(0)$, *and* $v_5=(4)$. *If* card$(N(14)\cap$ $\cap V(C))=d-2$ *and* $(134)\in V(C)$, *then*

$$G(v_3)\geqq\frac{d-4}{d-2}+\frac{2}{d-3}.$$

Proof. Putting $M=\{5,6,\dots,d\}$ we have $N(14)=\{(1),(124),(134),(4)\}\cup\{(14i)\colon i\in M\}$. Because C is of spread 2, we get $(124)\notin V(C)$ and $(3)\notin V(C)$. Let $(14m)$

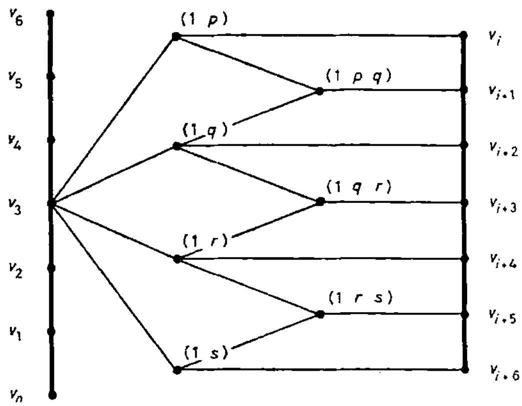

Fig. 3

be the second element of $N(14)$ besides (124) not lying in $V(C)$, $(m \in M)$. If some element $(13i)$ with $i \in M$ is not in $V(C)$, we easily can conclude in the way we already used that

1) for $i=m$, $G(v_3) \geq \frac{d-3}{d-2}+\frac{1}{d-4} > \frac{d-4}{d-2}+\frac{2}{d-3}$

2) for $i \neq m$, $G(v_3) \geq \frac{d-4}{d-2}+\frac{2}{d-3}$.

Also, if some element $(1ij)$ with $i, j \in M$ and $i \neq j$ is not in $V(C)$, we get

1) for $i=m$ or $j=m$, $G(v_3) \geq \frac{1}{d-1}+\frac{d-5}{d-2}+\frac{1}{d-3}+\frac{1}{d-4} > \frac{d-4}{d-2}+\frac{2}{d-3}$

2) for $i \neq m$ and $j \neq m$, $G(v_3) \geq \frac{1}{d-1}+\frac{d-6}{d-2}+\frac{3}{d-3} > \frac{d-4}{d-2}+\frac{2}{d-3}$.

In all these cases the assertion of the lemma is true.

In the following we assume that, for all $i, j \in M$, $(13i) \in V(C)$ and $(1ij) \in V(C)$. Since $d \geq 7$, we know that there are at least three elements $(13i)$ in $V(C)$ $(i \in M)$. Because for all $i \in M$, $d_I((13i), v_5)=4$, there exist two different elements $p, q \in M$ with $d_C((13p), v_1), v_1) \geq 4$, $d_C((13p), v_5) \geq 4$, $d_C((13q), v_1) \geq 4$, and $d_C((13q), v_5) \geq 4$. Without loss of generality we assume $p \neq m$. We inspect $y \in N(13p) \cap V(C)$. If $y \in \{(13), (1p), (123p)\}$ we come to the same contradictions as in the proof of lemma 1. $y=(13pr)$ with $r \in M \setminus \{p\}$ leeds to $d_I(y, (13r))=1$ and $d_I(y, (1pr))=1$. Thus $(1pr) \notin V(C)$ or $(13r) \notin V(C)$. In these cases the lemma is true as shown above. Thus $(13p)$ has the neighbours $(3p)$ and $(134p)$ in C. Since $(134) \in V(C)$ and C is of spread 2, $(134p)$ has the neighbour (134) in C. Then $(14p)$ cannot belong to C. Since $p \neq m$ we get a contradiction to our assumption that all elements of $N(14)$ except (124) and $(14m)$ belong to $V(C)$. Thus the above assumption is false, and lemma 3 is proved. ∎

Lemma 4. *Let d and C be as in lemma 3.*

If $\operatorname{card}(N(14)\cap V(C))=d-2$ *and* $(134)\notin V(C)$, *then*

$$G(v_3) \geqq \frac{d-4}{d-2}+\frac{2}{d-3}.$$

Proof. Let $M=\{5,6,\ldots,d\}$. Since $(124)\in Sv_2, (134)\notin V(C)$, and $\operatorname{card}(N(14)\cap \cap V(C))=d-2$, we know that for all $i\in M$ $(14i)$ is lying in $V(C)$. $(13i)\notin V(C)$ for $i\in M$ or $(1ij)\notin V(C)$ for $i, j\in M$ with $i\neq j$ leads in the usual way to $G(v_3)\geqq \geqq\frac{d-4}{d-2}+\frac{2}{d-3}$, and in these cases the lemma is proved.

In the following we will show that C cannot fulfil the conditions

(i) $N(13)\setminus V(C)=\{(3),(134)\}$,
(ii) $N(14)\setminus V(C)=\{(124),(134)\}$,
(iii) $N(1i)\setminus V(C)=\{(i),(12i)\}$ for all $i\in M$,

and so one of the above cases is valid.

We assume C to fulfil all these conditions (i) to (iii). Then for all i, $j\in M$ with $i\neq j$ $(1ij)$ is in $V(C)$. Necessarily $(1ij)$ has the neighbours (ij) and $(12ij)$ in C. That means

(4.1.) *For all i, $j\in M$ with $i\neq j$ the graph of figure 4 is a subgraph of C.*

Now we need the following fact, namely

(4.2.) *For all $i\in M$ $(123i)\neq v_n$ holds.*

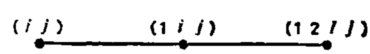

Fig. 4

This we proof indirectly by assuming $v_n=(123i)$ for one $i\in M$. This leads to $v_{n-1}=(13i)$ because of the condition (i) and the spread 2 of C. Since $d\geqq 7$ there exist $j, k\in M\setminus\{i\}$ with $j\neq k$. With (4.1.) we know that the graphs of figure 5 are subgraphs of C. The only possibility for a second neighbour of $(12ij)$ in C is $(124ij)$, and $(12ik)$ has the neighbour $(124ik)$ in C. With condition (ii) we have $(14i)\in V(C)$. Now we try to find the neighbours of $(14i)$ in C. We have $N(14i)= =\{(14),(1i),(4i),(124i),(134i)\}\cup\{(14is)\colon s\in M\setminus\{i\}\}$. Let $y\in N(14i)\cap V(C)$. If $y=(14)$ or $y=(1i)$, then $d_I(y, v_3)=1$ and $d_C(y, v_3)\geqq 5$, a contradiction to the spread 2 of C. If $y=(124i)$, then $d_I(y,(124ij))=1$ and $d_I(y,(124ik))=1$, hence $(124ik)\notin V(C)$ or $(124ij)\notin V(C)$, which is a contradiction, since the graphs of figure 5 should be subgraphs of C. If $y=(14is)$, then $d_I(y,(1is))=1$ and $d_I(y,(14s))=1$, hence $(14s)\notin V(C)$ or $(1is)\notin V(C)$, a contradiction to condition (i) or (iii). Thus $(14i)$ has the neighbours $(4i)$ and $(134i)$ in C. Since C is of spread

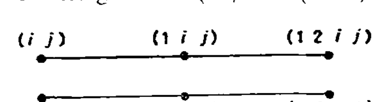

Fig. 5

2, C is the graph shown in figure 6, and all of the conditions (i) to (iii) are hurt. Thus (4.2.) is proved.

With help of (4.2.) we conclude that for all $i\in M$ $(13i)\in V(C)$ has the neighbours $(3i)$ and $(134i)$ in C, and $(3i)$ has $(23i)$ as second neighbour in C. That means

(4.3.) *For all $i\in M$ the graph of figure 7 is a subgraph of C.*

This helps us to see, that for all $i, j\in M$ with $i\neq j$ (ij) has the neighbour $(4ij)$ in C, and we get with (4.1.).

(4.4.) *For all $i, j\in M$ with $i\neq j$ the graph of figure 8 is a subgraph of C.*

If some element $(14i)\in V(C)$ had the neighbour $(4i)$ in C, we would get, since C is of spread 2, $v_6=(4i)$ and $v_7=(14i)$. Then for no $j\in M\setminus\{i\}$ (ij) could have the neighbour $(4ij)$ in C. This would be a contradiction to (4.4.). Thus no element $(14i)$ has the neighbour $(4i)$ in C, and we can conclude in the usual way (4.5.)

(4.5.) *For all $i\in M$ the graph of figure 9 is a subgraph of C.*

We have $N(23i)=\{(23),(2i),(3i),(123i),(234i)\}\cup\{(23ij)\colon j\in M\setminus\{i\}\}$. Let $y\in N(23i)\cap V(C)$. If $y=(2i)$ or $y=(234i)$, then $d_I(y,(24i))=1$ and $d_C(y,(24i))=7$, this is impossible. If $y=(123i)$, then $d_I(y,v_1)=1$, hence $v_n=y$, a contradiction to (4.2.). If $y=(23)$, then $d_I(y,v_1)=1$, hence $v_1=y$ and for all $s\in M\setminus\{i\}$ $(23s)$ cannot be in $V(C)$, this is a contradiction to (4.3.). Thus for all $i\in M$ $(23i)$ has the

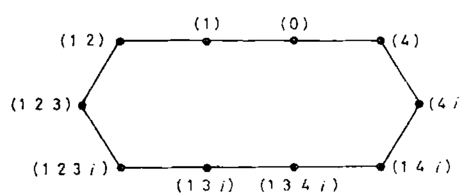

Fig. 6

Fig. 7

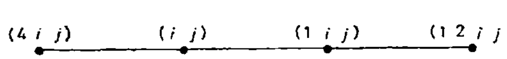

Fig. 8

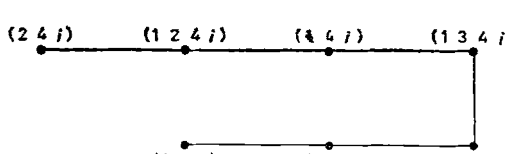

Fig. 9

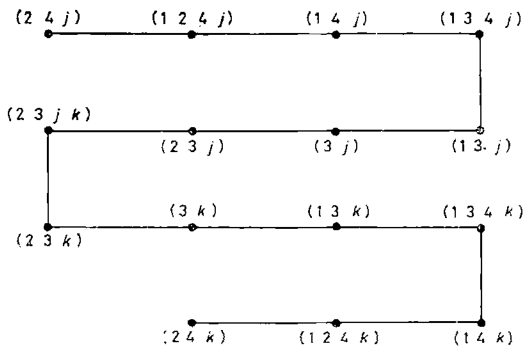

Fig. 10

neighbour $(23ik)$ in $C, (k \in M \setminus \{i\})$. Now we choose j and k in M so that $(23j)$ has the neighbours $(3j)$ and $(23jk)$ in C. Then the following holds:

(4.6.) *The graph of figure 10 is a subgraph of C.*

Now we remember (4.1.) and try to find a second neighbour of $(12jk)$ in C. We have $N(12jk) = \{(12j), (12k), (1jk), (2jk), (123jk), (124jk)\} \cup \{(12jkm): m \in M \setminus \{j, k\}\}$. Let $y \in N(12jk) \cap V(C)$. If $y = (12j)$ or $y = (12k)$, then $d_I(y, v_2) = 1$ and $d_C(y, v_2) \geqq 3$. If $y = (2jk)$ or $y = (123jk)$, then $d_I(y, (23jk)) = 1$ and with (4.6.) $d_C(y, (23jk)) \geqq 8$. If $y = (124jk)$, then $d_I(y, (124j)) = 1$ and with (4.4.) $d_C(y, (124j)) \geqq 3$. All these cases are impossible since C is of spread 2. If $y = (12jkm)$, then $d_I(y, (12jm)) = 1$ and $d_I(y, (12km)) = 1$, hence $(12km) \notin V(C)$ or $(12jm) \notin V(C)$, a contradiction to (4.1.). Thus $(12jk)$ has only the neighbour $(1jk)$ in C, this is a contradiction since C is a circuit, and lemma 4 is proved. ∎

Lemma 5. *Let d and C be as in lemma 3.*
If card$(N(14) \cap V(C)) = d-1$, *then*

$$G(v_3) > \frac{d-4}{d-2} + \frac{2}{d-3}.$$

Proof. Since $(124) \in Sv_2$, $(124) \notin V(C)$. All other elements of $N(14)$ lie in $V(C)$. Putting $M = \{3, 5, 6, \dots, d\}$ we have $N(14) = \{(1), (4), (124)\} \cup \{(14i): i \in M\}$. It is $N(14) \cap \{v_1, \dots, v_5\} = \{(1), (4)\}$. Set $j = \min \{i \in \mathbf{N}: 6 \leqq i \leqq n$ and $v_i \in N(14)\}$. $v_j = (14p)$ with $p \in M$ and $v_{j+1} = (124p)$ or $v_{j+1} = (14pq)$ with $q \in M \setminus \{p\}$.

Case 1: $v_{j+1} = (14pq)$ with $q \in M \setminus \{p\}$.
Then $v_{j+2} = (14q)$ and $v_{j+3} = (124q)$ or $v_{j+3} = (14qr)$ with $r \in M \setminus \{p, q\}$.

Case 1.1.: $v_{j+3} = (14qr)$ with $r \in M \setminus \{p, q\}$.

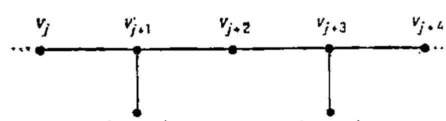

Fig. 11

Then $v_{j+4}=(14r)$, and we have the situation of figure 11. We get

$$G(v_3) \geqq \frac{2}{d-1}+\frac{d-7}{d-2}+\frac{2}{d-3}+\frac{1}{d-4} \quad \text{if} \quad 3 \notin \{p, q, r\},$$

$$G(v_3) \geqq \frac{1}{d-1}+\frac{d-5}{d-2}+\frac{1}{d-3}+\frac{1}{d-4} \quad \text{if} \quad 3 \in \{p, r\},$$

$$G(v_3) \geqq \frac{1}{d-1}+\frac{d-6}{d-2}+\frac{3}{d-3} \quad \text{if} \quad 3 = q.$$

Since $\frac{2}{d-1}+\frac{d-7}{d-2}+\frac{2}{d-3}+\frac{1}{d-4}>\frac{1}{d-1}+\frac{d-5}{d-2}+\frac{1}{d-3}+\frac{1}{d-4}>\frac{1}{d-1}+\frac{d-6}{d-2}+$ $+\frac{3}{d-3}>\frac{d-4}{d-2}+\frac{2}{d-3}$, we have proved the assertion of lemma 5 in thes case 1.1.

Case 1.2.: $v_{j+3}=(124q)$.

Then $v_{j+4}\notin N(14)$. Since $d\geqq 7$ there exists at least one element of $N(14)$ in $\{v_{j+5}, \dots, v_n\}$. Set $k=\min\{i\in \mathbf{N}: j+5\leqq i\leqq n$ and $v_i\in N(14)\}$, then $v_k=(14r)$ with $r\in M\setminus\{p,q\}$, $v_{k-1}=(124r)$, and $v_{k+1}=(12rs)$ with $s\in M\setminus\{p,q,r\}$. Then $v_{k+2}=(14s)$, and we have the situation of figure 12. We get

$$G(v_3) \geqq \frac{2}{d-1}+\frac{d-8}{d-2}+\frac{4}{d-3} \quad \text{if} \quad 3 \notin \{p, q, r, s\},$$

$$G(v_3) \geqq \frac{1}{d-1}+\frac{d-6}{d-2}+\frac{3}{d-3} \quad \text{if} \quad 3 \in \{p, q, r, s\}.$$

Since $\frac{2}{d-1}+\frac{d-8}{d-2}+\frac{4}{d-3}>\frac{1}{d-1}+\frac{d-6}{d-2}+\frac{3}{d-3}>\frac{d-4}{d-2}+\frac{2}{d-3}$ we have proved the assertion of lemma 5 in case 1.2.

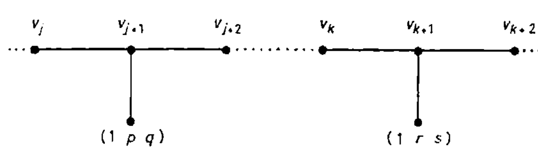

Fig. 12

Case 2: $v_{j+1}=(124p)$.

Then $v_{j+2}\notin N(14)$. Set $k=\min\{i\in\mathbf{N}: j+3\leqq i\leqq n$ and $v_i\in N(14)\}$ then $v_k=(14q)$ with $q\in M\setminus\{p\}$, $v_{k-1}=(124q)$, and $v_{k+1}=(14qr)$ with $r\in M\setminus\{p,q\}$. Hence $v_{k+2}=$ $=(14r)$, and we get the same bounds as in case 1 by distinguishing

Case 2.1.: $v_{k+3}=(14rs)$ with $s\in M\setminus\{p,q,r\}$

and

Case 2.2.: $v_{k+3}=(124r)$. ∎

3. The upper bound

Now we are able to prove the central theorem of this paper.

Theorem 1. $C(d,2) \leqq 2^{d-1} - \dfrac{2^{d-1}}{d(d-5)+7}$ *for* $d \geqq 7$.

Proof. Let C be a maximal d-dimensional circuit code of spread 2. Denote by (2) the following statement

(2) "For all $v_i \in V(C)$ the inequality $G(v_i) \geqq \dfrac{d-4}{d-2} + \dfrac{2}{d-3}$ holds."

If (2) holds, theorem 1 is true as one can see from the following:

$$2^d - C(d,2) \overset{(1)}{\geqq} \sum_{x \in V(C)} G(x) \overset{(2)}{\geqq} C(d,2)\left(\frac{d-4}{d-2} + \frac{2}{d-3}\right).$$

That means $2^d \geqq C(d,2)\left(\dfrac{2d^2-10d+14}{d^2-5d+6}\right)$ or equivalently

$$C(d,2) \leqq 2^{d-1} - \frac{2^{d-1}}{d(d-5)+7}.$$

So we only have to prove (2). Let $v_i \in V(C)$. There exists a symmetry of $I(d)$ carrying the triple (v_{i-1}, v_i, v_{i+1}) onto the triple $((12),(1),(0))$. Thus without loss of generality we can assume $C = (v_1, v_2, \ldots, v_n; v_1)$ with $n = C(d,2)$ and $v_2 = (12)$, $v_3 = (1)$, and $v_4 = (0)$. Since C is of spread 2, $v_1 \neq (1)$ and $v_1 \neq (2)$. So, by eventually changing coordinates, we have $v_1 = (123)$, and we only have to show

$$G(v_3) \geqq \frac{d-4}{d-2} + \frac{2}{d-3}. \tag{3}$$

Since C is of spread 2, $v_5 \neq (1)$ and $v_5 \neq (2)$. We must distinguish whether $d_I(v_1, v_5) = 2$ or $d_I(v_1, v_5) = 4$ holds, that means $v_5 = (3)$ or (eventually by changing coordinates) $v_5 = (4)$.

Case 1: $v_5 = (3)$

Since (123), (1), and (3) are in $V(C)$, we have $\operatorname{card}(N(13) \cap V(C)) \geqq 3$. If exactly s elements of $N(13)$ are in $V(C)$ with $3 \leqq s \leqq d$, then we get $g_{v_3}((13)) = \dfrac{1}{s}$. Then $d-s$ elements $(13i)$ of $N(13)$ with $i \in M = \{4, 5, \ldots, d\}$ are not in $V(C)$, and we have for these $i \in M$ $g_{v_3}((1i)) \geqq \dfrac{1}{d-3}$, since $(12i) \in Sv_2$, $(i) \in Sv_4$, and $(13i) \notin V(C)$. For the other $s-3$ elements $i \in M$ we get $g_{v_3}((1i)) \geqq \dfrac{1}{d-2}$, since $(12i) \in Sv_2$ and $(i) \in Sv_4$. Hence $G(v_3) \geqq \dfrac{1}{s} + \dfrac{s-3}{d-s} + \dfrac{d-s}{d-3} = Z_1(s)$ for $3 \leqq s \leqq d$, if $\operatorname{card}(N(13) \cap V(C)) = s$. With $Z_1(s) > Z_1(s+1)$ and $Z_1(d-2) = \dfrac{d-4}{d-2} + \dfrac{2}{d-3}$ the proof of (3) is done for $3 \leqq s \leqq d-2$. The cases $s = d-1$ and $s = d$ are done in lemma 1 and lemma 2.

Case 2: $v_5=(4)$

Now we will concentrate on $N(14)$. Since $(124)\in Sv_2$ and $(1),(4)\in V(C)$, $2\leqq \text{card}\,(N(14)\cap V(C))\leqq d-1$. We assume $\text{card}\,(N(14)\cap V(C))=s$ with $2\leqq s\leqq$ $\leqq d-1$. Then $g_{v_3}((14))=\frac{1}{s}$.

If (134) belongs to these s elements, that means if $(134)\in V(C)$ (then $s\geqq 3$), then $d-s-1$ elements $(14i)$ with $i\in N=\{5,6,\ldots,d\}$ are not in $V(C)$, and we get for these $i\in N$, $g_{v_3}((1i))\geqq\frac{1}{d-3}$, since $(12i)\in Sv_2$, $(i)\in Sv_4$, and $(14i)\notin V(C)$. For the other $s-3$ elements $i\in N$ we have $g_{v_3}((1i))\geqq\frac{1}{d-2}$, since $(12i)\in Sv_2$ and $(i)\in Sv_4$. Hence $G(v_3)\geqq\frac{1}{s}+\frac{1}{d-1}+\frac{s-3}{d-2}+\frac{d-s-1}{d-3}=Z_2(s)$ for $3\leqq s\leqq d-1$.

If (134) does not belong to these s elements (that means $(134)\notin V(C)$ and necessarily $s\leqq d-2$), then $d-s-2$ elements $(14i)$ with $i\in N$ are not in $V(C)$ and we get

$$G(v_3)\geqq\frac{1}{s}+\frac{1}{d-2}+\frac{s-2}{d-2}+\frac{d-s-2}{d-3}=Z_3(s)\quad\text{for}\quad 2\leqq s\leqq d-2.$$

Obviously the following relations hold: $Z_2(s)>Z_2(s+1)$, $Z_3(s)>Z_3(s+1)$, $Z_2(s)>$ $>Z_3(s)$, and $Z_3(d-3)=\frac{d-4}{d-2}+\frac{2}{d-3}$. Thus (3) is proved if $\text{card}\,(N(14)\cap V(C))\geqq$ $\leqq d-3$. The remaining cases

$\text{card}\,(N(14)\cap V(C))=d-2$ and $(134)\in V(C)$
$\text{card}\,(N(14)\cap V(C))=d-2$ and $(134)\notin V(C)$
$\text{card}\,(N(14)\cap V(C))=d-1$

are done in the lemmas 3 to 5. Thus (3) holds in all cases, and theorem 1 is proved. ∎

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